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BASINS OF ATTRACTION FOR OPTIMAL EIGHTH ORDER METHODS TO FIND SIMPLE ROOTS OF NONLINEAR EQUATIONS

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Abstract. Several optimal eighth order methods to obtain simple roots are analyzed. The methods are based on two step, fourth order optimal methods and a third step of modified Newton. The modification is performed by taking an interpolating polynomial to replace either $f(z_n)$ or $f'(z_n)$. In six of the eight methods we have used a Hermite interpolating polynomial. The other two schemes use inverse interpolation. We discovered that the eighth order methods based on Jarratt's optimal fourth order methods perform well and those based on King's or Kung-Traub's methods do not. In all cases tested, the replacement of $f(z)$ by Hermite interpolation is better than the replacement of the derivative, $f'(z)$.

Keywords: Basin of Attraction, optimal methods, simple roots, nonlinear equations, interpolation

1. Introduction

A vast number of different methods have been proposed for the numerical solution of nonlinear equations. The methods are classified by their order of convergence, p, and the number, d, of function- (and derivative-) evaluation per step. There are two efficiency measures (see Traub [1]) defined as $I = p/d$ (informational efficiency) and $E = p^{1/d}$ (efficiency index). Another measure, introduced recently, is the basin of attraction. See Stewart [2], Scott et al [3], Amat et al [4], [5], [6], [7], Chicharro et al. [8], Chun et al [9], Cordero et al. [10], Neta et al. [11], Gutiérrez et al. [12] and for methods to find multiple roots, see Neta et al [13].

In 1974, Kung and Traub [14] introduced the notion of optimality. They conjectured that multipoint methods without memory requiring $d + 1$ function-evaluations have order of convergence at most 2^d. Such methods are usually called optimal (see, for example, [15]). An optimal method of order $p = 2$ is the well known Newton's method. It was discussed by Stewart [2] and Scott et al [3] and thus will not be given here. Optimal methods of order four were discussed in [7], [9] and [11]. We have seen that the best fourth order method is due to Jarratt [16].

In this paper we develop and compare several new optimal methods of order eight. Using the techniques given by Petković et al. [15], the eighth order methods have been constructed by using optimal fourth order methods followed by a step of interpolation. Two different forms of the interpolation have been investigated. One where the interpolating polynomial replaces the function and one where the derivative is replaced. Two of the compared schemes use inverse interpolation [18].

In the next section we describe the methods to be considered in this comparative study. Section 3 will give the conjugacy maps for each method and find the extraneous fixed points (see [17].) We will show the relationship between these maps, the extraneous fixed points and the basins of attraction in our numerical experiments detailed in Section 4.

2. Methods for the comparative study

First, we list the eight eighth-order methods we consider here.

Petković et al. [15] have constructed eighth order methods using any optimal fourth order method followed by a step of interpolation. In the first two methods this idea was combined with Jarratt's optimal fourth order method [16] to create an optimal eighth order scheme. In other methods we used inverse interpolation.

–I– In the first version, denoted by JHID8, we added a Newton-like sub-step and replaced the derivative with a Hermite interpolating polynomial. The resulting scheme is of order eight. The method is given by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}u_n \\
t_n &= x_n - \frac{1}{2}u_n - \frac{1}{2}\frac{u_n}{1+\frac{3}{2}\left(\frac{f'(y_n)}{f'_n}-1\right)} \\
x_{n+1} &= t_n - \frac{f(t_n)}{H'_3(t_n)}
\end{aligned} \tag{1}
$$

where

$$
u_n = \frac{f_n}{f'_n}, \tag{2}
$$

and

$$
H'_3(t_n) = 2(f[x_n,t_n] - f[x_n,y_n]) + f[y_n,t_n] + \frac{y_n - t_n}{y_n - x_n}(f[x_n,y_n] - f'_n). \tag{3}
$$

–II– The second version denoted JHIF8 where the interpolating polynomial replacing the function (instead of derivative) at the third sub-step is given by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}u_n \\
t_n &= x_n - \frac{1}{2}u_n - \frac{1}{2}\frac{u_n}{1+\frac{3}{2}\left(\frac{f'(y_n)}{f'_n}-1\right)} \\
x_{n+1} &= t_n - \frac{H_3(t_n)}{f'(t_n)}
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
H_3(t_n) \quad &= f_n + f'_n \frac{(t_n - y_n)^2(t_n - x_n)}{(y_n - x_n)(x_n + 2y_n - 3t_n)} + f'(t_n)\frac{(t_n - y_n)(x_n - t_n)}{x_n + 2y_n - 3t_n} \\
&- \quad f[x_n,y_n]\frac{(t_n - x_n)^3}{(y_n - x_n)(x_n + 2y_n - 3t_n)}.
\end{aligned} \tag{5}
$$

–III– The next one is using Kung-Traub optimal fourth order [15] and Hermite interpolating polynomial. This is denoted HKT.

$$
(6) \qquad \begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f(y_n)}{f'_n}\frac{1}{[1 - f(y_n)/f_n]^2} \\
x_{n+1} &= t_n - \frac{f(t_n)}{H'_3(t_n)},
\end{aligned}
$$

where $H'_3(t_n)$ is given by (3).

–IV– The fourth method is using Hermite interpolating polynomial with King's fourth order method. This is denoted HK8.

$$
(7) \qquad \begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f(y_n)}{f'_n}\frac{f_n + \beta f(y_n)}{f_n + (\beta - 2)f(y_n)} \\
x_{n+1} &= t_n - \frac{H_3(t_n)}{f'(t_n)}
\end{aligned}
$$

where $H_3(t_n)$ is given by (5).

In our experiments we have used $\beta = 3 - 2\sqrt{2}$ which is the optimal parameter for King's method (see [11]).

–V– Next we took Kung-Traub's eighth order (KT8) method [14] based on inverse interpolation [18]. It is given by

$$
(8) \qquad \begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f_n}{f'_n}\frac{f(y_n)f_n}{[f_n - f(y_n)]^2} \\
x_{n+1} &= t_n - \frac{f_n}{f'_n}\frac{f_n f(y_n) f(t_n)}{[f_n - f(y_n)]^2}\frac{f_n^2 + f(y_n)\left[f(y_n) - f(t_n)\right]}{[f_n - f(t_n)]^2\left[f(y_n) - f(t_n)\right]},
\end{aligned}
$$

where $f_n = f(x_n)$ and similarly for the derivative.

–VI– Neta's eighth order (N8) method [19] is also based on inverse interpolation and given by

$$
(9)\qquad
\begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f(y_n)}{f'_n}\frac{f_n + \beta f(y_n)}{f_n + (\beta - 2)f(y_n)} \\
x_{n+1} &= x_n - u_n + \gamma f_n^2 - \rho f_n^3,
\end{aligned}
$$

where

$$
(10)\qquad
\begin{aligned}
\rho &= \frac{\phi_y - \phi_t}{F_y - F_t},\ \gamma = \phi_y - \rho F_y,\ F_y = f(y_n) - f_n,\ F_t = f(t_n) - f_n, \\
\phi_y &= \frac{y_n - x_n}{F_y^2} - \frac{1}{F_y f'_n},\ \phi_t = \frac{t_n - x_n}{F_t^2} - \frac{1}{F_t f'_n}.
\end{aligned}
$$

In our experiments we have used $\beta = 3 - 2\sqrt{2}$ which is the optimal parameter for King's method (see [11]). This is different from method HK8 in that it is using inverse interpolation instead of Hermite interpolating polynomial.

–VII– The seventh scheme considered is due to Wang and Liu [21]. Here we have the original method denoted by WL

$$
(11)\qquad
\begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f(y_n)}{f'_n}\frac{f_n}{f_n - 2f(y_n)} \\
x_{n+1} &= t_n - \frac{f(t_n)}{H'_3(t_n)}
\end{aligned}
$$

where $H'_3(t_n)$ is defined by (3). Note that the first two substeps are Ostrowski's method [22].

–VIII– The last scheme, denoted WLN, is similar to the seventh scheme except we replaced the function in the last sub-step by the Hermite polynomial instead of replacing the derivative.

$$
\begin{aligned}
y_n &= x_n - u_n \\
t_n &= y_n - \frac{f(y_n)}{f'_n}\frac{f_n}{f_n - 2f(y_n)} \qquad (12)\\
x_{n+1} &= t_n - \frac{H_3(t_n)}{f'(t_n)}
\end{aligned}
$$

where $H_3(t_n)$ is given by (5).

We will show in all cases tested, the replacement of $f(z)$ by Hermite interpolation is better than the replacement of the derivative, $f(z)$.

3. Corresponding Conjugacy Maps for Quadratic polynomials

Theorem 3.1. *(Hermite based Jarratt optimal eighth order methods, JHID8 and JHIF8) For a rational map $R_p(z)$ arising from the method (1) or (4) applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^8.$$

Theorem 3.2. *(Hermite based Kung-Traub eighth order optimal method, HKT) For a rational map $R_p(z)$ arising from the method (6) applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^8 \frac{z^4 + 4z^3 + 8z^2 + 8z + 4}{4z^4 + 8z^3 + 8z^2 + 4z + 1}.$$

Theorem 3.3. *(Hermite based Neta's optimal eighth order method, HK8) For a rational map $R_p(z)$ arising from the method (7) applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^8 \frac{z^4 + (2\beta+4)z^3 + (\beta^2+8\beta+6)z^2 + (4\beta^2+10\beta+4)z + (4\beta^2+4\beta+1)}{(4\beta^2+4\beta+1)z^4 + (4\beta^2+10\beta+4)z^3 + (\beta^2+8\beta+6)z^2 + (4+2\beta)z + 1}.$$

and for $\beta = 3 - 2\sqrt{2}$

(13)

$$\begin{aligned} N_z &= -z^4 + (-10 + 4\sqrt{2})z^3 + (-47 + 28\sqrt{2})z^2 + (-102 + 68\sqrt{2})z - 81 + 56\sqrt{2} \\ D_z &= (-81 + 56\sqrt{2})z^4 + (-102 + 68\sqrt{2})z^3 + (-47 + 28\sqrt{2})z^2 + (-10 + 4\sqrt{2})z - 1. \end{aligned}$$

Theorem 3.4. *(Kung-Traub's optimal eighth order method, KT8) For a rational map* $R_p(z)$ *arising from the method (8) applied to* $p(z) = (z - a)(z - b),\ a \neq b$, $R_p(z)$ *is conjugate via the Möbius transformation given by* $M(z) = \dfrac{z - a}{z - b}$ *to*

$$S(z) = z^8 \frac{N_z}{D_z}$$

where

$$\begin{aligned} N_z &= z^{16} + 10z^{15} + 52z^{14} + 182z^{13} + 479z^{12} + 1006z^{11} + 1749z^{10} \\ &\quad +2568z^9 + 3214z^8 + 3432z^7 + 3116z^6 + 2382z^5 + 1506z^4 \\ &\quad +760z^3 + 289z^2 + 74z + 10 \\ D_z &= 10z^{16} + 74z^{15} + 289z^{14} + 760z^{13} + 1506z^{12} + 2382z^{11} \\ &\quad +3116z^{10} + 3432z^9 + 3214z^8 + 2568z^7 + 1749z^6 + 1006z^5 \\ &\quad +479z^4 + 182z^3 + 52z^2 + 10z + 1. \end{aligned} \tag{14}$$

Theorem 3.5. *(Neta's optimal eighth order method, N8) For a rational map* $R_p(z)$ *arising from the method (9) applied to* $p(z) = (z-a)(z-b),\ a \neq b$, $R_p(z)$ *is conjugate via the Möbius transformation given by* $M(z) = \dfrac{z - a}{z - b}$ *to*

$$S(z) = z^8 \frac{N_z}{D_z}$$

where

$$
\begin{aligned}
N_z &= z^{16} + (10+3\beta)z^{15} + (3\beta^2+30\beta+49)z^{14} \\
&+(\beta^3+30\beta^2+144\beta+158)z^{13} + (10\beta^3+141\beta^2+450\beta+380)z^{12} \\
&+(46\beta^3+426\beta^2+1040\beta+732)z^{11} \\
&+(134\beta^3+943\beta^2+1904\beta+1180)z^{10} \\
&+(283\beta^3+1630\beta^2+2872\beta+1630)z^{9} \\
&+(458\beta^3+2269\beta^2+3644\beta+1945)z^{8} \\
&+(576\beta^3+2576\beta^2+3919\beta+2004)z^{7} \\
&+(558\beta^3+2394\beta^2+3566\beta+1778)z^{6} \\
&+(406\beta^3+1810\beta^2+2719\beta+1350)z^{5} \\
&+(212\beta^3+1085\beta^2+1704\beta+861)z^{4} \\
&+(69\beta^3+486\beta^2+848\beta+442)z^{3} \\
&+(10\beta^3+143\beta^2+316\beta+169)z^{2} + (20\beta^2+79\beta+42)z + (5+10\beta)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
D_z &= (10\beta+5)z^{16} + (20\beta^2+79\beta+42)z^{15} \\
&+(10\beta^3+143\beta^2+316\beta+169)z^{14} \\
&+(69\beta^3+486\beta^2+848\beta+442)z^{13} \\
&+(212\beta^3+1085\beta^2+1704\beta+861)z^{12} \\
&+(406\beta^3+1810\beta^2+2719\beta+1350)z^{11} \\
&+(558\beta^3+2394\beta^2+3566\beta+1778)z^{10} \\
&+(576\beta^3+2576\beta^2+3919\beta+2004)z^{9} \\
&+(458\beta^3+2269\beta^2+3644\beta+1945)z^{8} \\
&+(283\beta^3+1630\beta^2+2872\beta+1630)z^{7} \\
&+(134\beta^3+943\beta^2+1904\beta+1180)z^{6} \\
&+(46\beta^3+426\beta^2+1040\beta+732)z^{5} \\
&+(10\beta^3+141\beta^2+450\beta+380)z^{4} \\
&+(\beta^3+30\beta^2+144\beta+158)z^{3} + (3\beta^2+30\beta+49)z^{2} \\
&+(3\beta+10)z + 1
\end{aligned} \tag{16}
$$

and for $\beta = 3 - 2\sqrt{2}$

$$
\begin{aligned}
N_z = & -z^{16} + (6\sqrt{2} - 19)z^{15} + (96\sqrt{2} - 190)z^{14} \\
& +(718\sqrt{2} - 1199)z^{13} + (3292\sqrt{2} - 5117)z^{12} \\
& +(-15648 + 10412\sqrt{2})z^{11} + (24504\sqrt{2} - 36189)z^{10} \\
& +(45114\sqrt{2} - 65973)z^{9} + (-96792 + 66576\sqrt{2})z^{8} \\
& +(79070\sqrt{2} - 114577)z^{7} + (74920\sqrt{2} - 108416)z^{6} \\
& +(-80471 + 55578\sqrt{2})z^{5} + (-45406 + 31268\sqrt{2})z^{4} \\
& +(12358\sqrt{2} - 18079)z^{3} + (-4538 + 3048\sqrt{2})z^{2} \\
& +(-619 + 398\sqrt{2})z - 35 + 20\sqrt{2} \\
D_z = & (-35 + 20\sqrt{2})z^{16} + (-619 + 398\sqrt{2})z^{15} + (-4538 + 3048\sqrt{2})z^{14} \\
& +(12358\sqrt{2} - 18079)z^{13} + (-45406 + 31268\sqrt{2})z^{12} \\
& +(-80471 + 55578\sqrt{2})z^{11} + (74920\sqrt{2} - 108416)z^{10} \\
& +(79070\sqrt{2} - 114577)z^{9} + (-96792 + 66576\sqrt{2})z^{8} \\
& +(45114\sqrt{2} - 65973)z^{7} + (24504\sqrt{2} - 36189)z^{6} \\
& +(-15648 + 10412\sqrt{2})z^{5} + (3292\sqrt{2} - 5117)z^{4} \\
& +(718\sqrt{2} - 1199)z^{3} + (96\sqrt{2} - 190)z^{2} + (6\sqrt{2} - 19)z - 1.
\end{aligned} \tag{17}
$$

Theorem 3.6. *(Wang-Liu eighth order optimal methods, WL and WLN) For a rational map* $R_p(z)$ *arising from the method (11) or (12) applied to* $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ *is conjugate via the Möbius transformation given by* $M(z) = \dfrac{z-a}{z-b}$ *to*

$$S(z) = z^8.$$

Note that the maps are of the form $S(z) = z^p R(z)$ where $R(z)$ is either unity or a rational function.

3.1. **Extraneous fixed points.** Note that all these methods can be written as

$$x_{n+1} = x_n - u_n H_f(x_n, y_n, t_n).$$

Clearly the root α is a fixed point of the method, since $u_n(\alpha) = 0$. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the method, since the second term on the right vanishes. These points are called extraneous fixed points (see [17]). The fixed point ξ is attractive, indifferent or repulsive depending on whether $|R_p'(\xi)|$ is less than, equal or greater than one, where $R_p(z) = z - u(z)H_f(z, y(z), t(z))$ is the iteration function.

Theorem 3.7. *The extraneous fixed points of Hermite based Jarratt's eighth-order method (1) are at* $z = 1.1504 \pm .53936i$, $z = .5782 \pm .36400i$, $z = -.015795 \pm .254898i$, $z = -.57950 \pm .05708$. *All fixed points are repulsive.*

The simple poles are at $z = -.56520$, $z = .78260 \pm .52171i$, $z = .71745 \pm .29499i$, $z = -.10446 \pm .50454i$, *and* $z = -.62598$.

Theorem 3.8. *The extraneous fixed points of Hermite based Jarratt's eighth-order method (4) are at* $z = .2282434731i$, $z = \pm 2.0765213397i$, *and* $z = \pm .7974733886i$. *All fixed points are repulsive.*

The simple poles are at $z = \pm\sqrt{3 \pm 2\sqrt{2}}i$, *and* $z = \pm i$.

Theorem 3.9. *The extraneous fixed points of Hermite based Kung-Traub's eighth-order method (HKT) are at* $z = -.48401 \pm .093413i$, $z = -.25752 \pm .37992i$, $z = -.19422 \pm .48532i$, $z = .21106 \pm .36453i$, $z = .26123 \pm .49043i$, $z = .36073$, $z = .40745 \pm .92157i$, *and at* $z = 4.89416$. *All fixed points are repulsive.*

The simple poles are at $z = -.59234$, $z = -.20254 \pm .45776i$, $z = .24924 \pm .38692i$, $z = .34385 \pm .89384i$, $z = 4.95411$, *and the double poles are at* $z = 0, \pm\frac{\sqrt{3}}{3}$.

Theorem 3.10. *The extraneous fixed points of HK8 (7) are at the roots of a polynomial* Q_{10} *of degree 10 in* z^2 *(assuming* $\beta = 3 - 2\sqrt{2}$*)*

$$
\begin{aligned}
&Q_{10}(z) = (-740\sqrt{2} + 1183)z^{10} + (501 + 68\sqrt{2})z^8 + (1176\sqrt{2} - 1386)z^6 \\
&+(-632\sqrt{2} + 906)z^4 + (140\sqrt{2} - 197)z^2 - 12\sqrt{2} + 17
\end{aligned}
$$

For $\beta = 3-2\sqrt{2}$ *we get the fixed points at* $z = \pm.166892805671862 \pm .175488988836070i$, $z = \pm 1.963305308625 13i$, $z = \pm .693658358342116i$, *and* $z = \pm .183870724371883i$.

The poles are at $z = \pm .9175962359i$, $z = \pm 2.305351882i$, $z = \pm .1159903203 \pm .2666600162i$, *and* $z = 0$. *The last one is of multiplicity 2.*

All fixed points are repulsive.

Theorem 3.11. *The extraneous fixed points of Kung-Traub's eighth-order method (KT8) are at the roots of a polynomial* Q_{22} *of degree 22 in* z^2

$$
\begin{aligned}
&Q_{22}(z) = 56239z^{22} + 281123z^{20} + 593633z^{18} + 617605z^{16} \\
&+355510z^{14} + 144926z^{12} + 38978z^{10} + 7850z^8 + 1131z^6 \\
&+143z^4 + 13z^2 + 1
\end{aligned}
$$

These extraneous fixed points are at $z = \pm.29669 \pm .22853i$, $z = \pm.33580 \pm .51558i$, $z = \pm.18588 \pm .38359i$, $z = \pm.19607 \pm .42724i$, $z = \pm.38347 \pm 1.30296i$, *and* $z = \pm 1.072134i$.

The poles are at $z = \pm 1.17799i$, and $z = \pm.23449 \pm .34932i$, $z = \pm 1.56402i$, $z = \pm.23194 \pm .43343i$ and $z = \pm\frac{\sqrt{3}}{3}i$. The first 6 are simple and the last 8 are double.

Theorem 3.12. *The extraneous fixed points of Neta's eighth-order method (N8) are at the roots of a polynomial Q_{10} of degree 5 in z^2 (assuming $\beta = 3 - 2\sqrt{2}$)*

$$Q_{10}(z) = 304289z^{10} + (693323 + 451184\sqrt{2})z^8 + (100842 + 365568\sqrt{2})z^6$$
$$+(136438 - 77216\sqrt{2})z^4 + (-25851 + 19840\sqrt{2})z^2 + 2351 - 1616\sqrt{2}$$

For $\beta = 3-2\sqrt{2}$ we get the fixed points at $z = \pm.166892799425929 \pm .175488993956276i$, $z = \pm.183870699530320i$, $z = \pm.693658359731125i$ and $z = \pm 1.96330530989740i$.

The poles are at $z = \pm.9175962359i$, $z = \pm 2.305351882i$ and $z = \pm.1159903203 \pm .2666600162i$.

All fixed points are repulsive.

Theorem 3.13. *There are* **no** *extraneous fixed points of Wang-Liu's eighth-order method (WL).*

Theorem 3.14. *The extraneous fixed points of second version of Wang-Liu's eighth-order method (WLN) are at $z = .2282434731i$, $z = \pm 2.0765213397i$, and $z = \pm.7974733886i$. All fixed points are repulsive.*

The simple poles are at $z = \pm\sqrt{3 \pm 2\sqrt{2}}i$, and $z = \pm i$. These are identical to those of method JHIF8

4. Numerical experiments

- Example 1
  In our first experiment, we have run all the methods to obtain the real simple zeros of the quadratic polynomial $z^2 - 1$. The results of the basins of attraction are given in Figures 1-8.
  Notice that the two methods based on Jarratt's method shown in Figure 1-2 and the modified Wang-Liu's method (WLN, Figure 8) perform best. Kung-Traub's method (Figure 5), Neta's method (Figure 6) and Wang-Liu's method (Figure 7) have black dots which means that the methods did not converge in 40 iterations starting at those points. Kung-Traub's method has regions along the imaginary axis, which are all solidly black. The second version of Wang-Liu (Figure 8) does not have the black dots, which we have seen in Wang-Liu's method.
- Example 2

In our next experiment we have taken the cubic polynomial $z^3 - 1$. The results are given in Figures 9-16. Again the results in Figures 9, 10 and 16 are best. The other methods are all having black regions.

- Example 3

  The results for the cubic polynomial $z^3 - z$ are given in Figures 17-24. The best methods are again JHID8 (Figure 17), JHIF8 (Figure 18) and WLN (Figure 24)

- Example 4

  Figures 25-32 show the results for the polynomial $z^4 - 10z^2 + 9$. Again the best results are using JHID8 (Figure 25), JHIF8 (Figure 26) and WLN (Figure 32). In this case even the original Wang-Liu (Figure 31) performed very well.

- Example 5

  The fifth order polynomial, $z^5 - 1$, results are shown in Figures 33-40. Here only JHIF8 (Figure 34) and WLN (Figure 40) perform best. All other methods suffer from slow convergence.

- Example 6

  The next example is for a polynomial of degree 6 with complex coefficients, $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$. The results are presented in Figures 41-48. The results are similar to Example 3.

- Example 7

  The last example for a polynomial of degree 7, $z^7 - 1$. The results are presented in Figures 49-56. In this case all methods have black dots. But the number of those is the smallest for all the methods JHID8, JHIF8 and WLN.

Conclusions We have produced several new eighth order methods by starting with some well-known fourth order methods and added a Newton-like third step. In that third step, we investigated replacing the derivative or the function with a Hermite interpolating polynomial. Two of the schemes (KT8 and N8) use inverse interpolation in the third sub-step. In all cases based on Hermite interpolating polynomials, we found the replacement of the function performed much better than by replacing the derivative. In addition, we found that the new eighth order Jarratt type methods and the modified Wang and Liu method performed the best while those methods based on King's method were the worst, even with the best choice of beta. Methods KT8 and N8, based on inverse interpolation, performed poorly in all seven examples.

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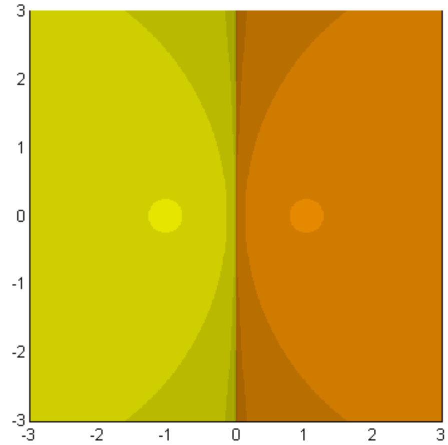

FIGURE 1. JHID8. The results are for the polynomial $z^2 - 1$

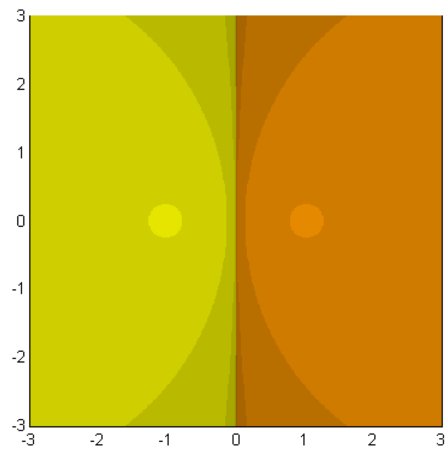

FIGURE 2. JHIF8. The results are for the polynomial $z^2 - 1$

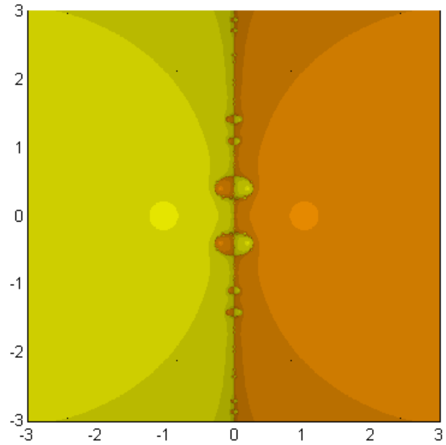

FIGURE 3. HKT. The results are for the polynomial $z^2 - 1$

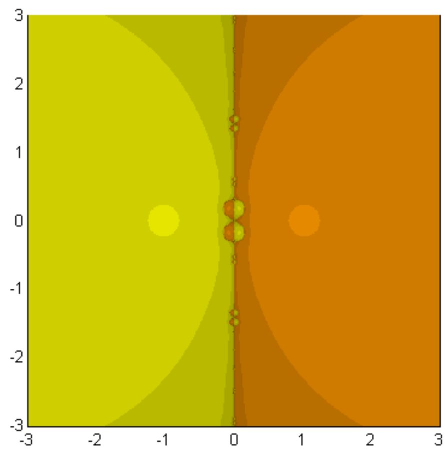

FIGURE 4. HK8. The results are for the polynomial $z^2 - 1$

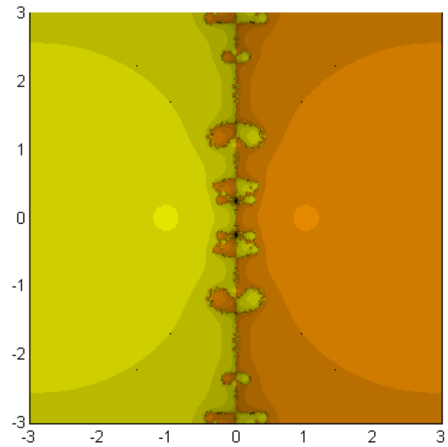

FIGURE 5. KT8. The results are for the polynomial $z^2 - 1$

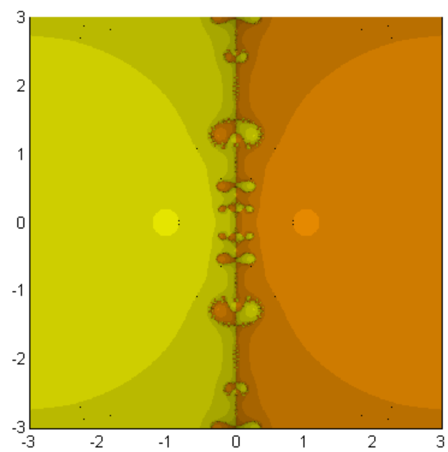

FIGURE 6. N8. The results are for the polynomial $z^2 - 1$

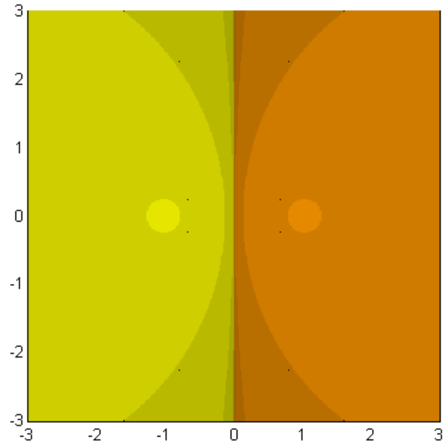

FIGURE 7. WL. The results are for the polynomial $z^2 - 1$

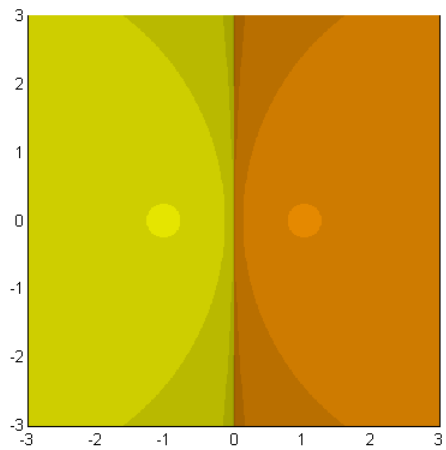

FIGURE 8. WLN. The results are for the polynomial $z^2 - 1$

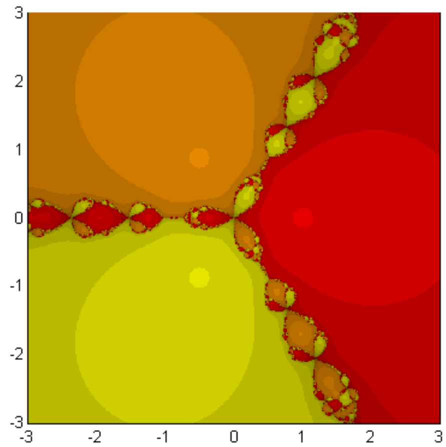

FIGURE 9. JHID8. The results are for the polynomial $z^3 - 1$

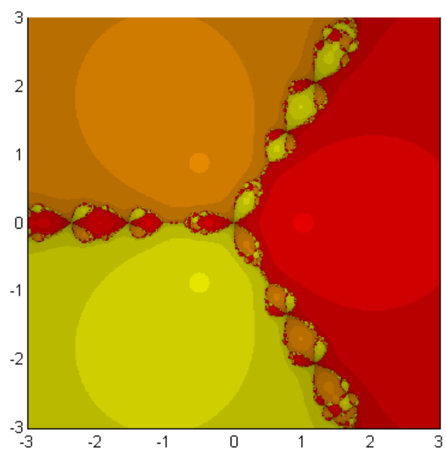

FIGURE 10. JHIF8. The results are for the polynomial $z^3 - 1$

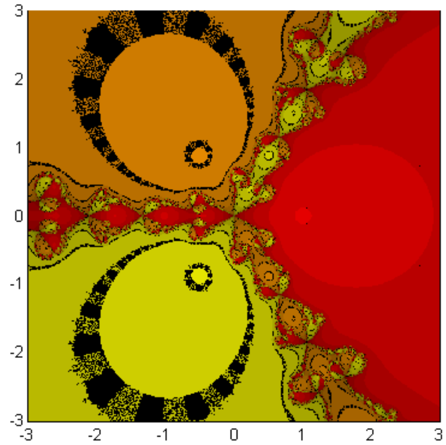

FIGURE 11. HKT. The results are for the polynomial $z^3 - 1$

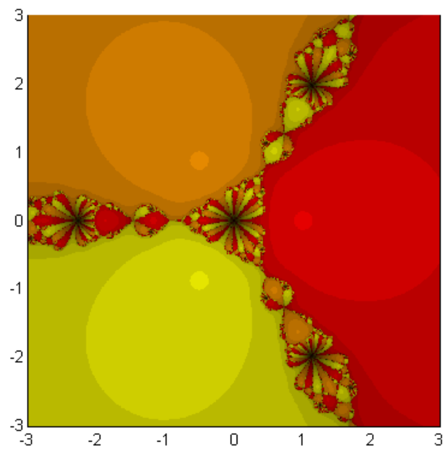

FIGURE 12. HK8. The results are for the polynomial $z^3 - 1$

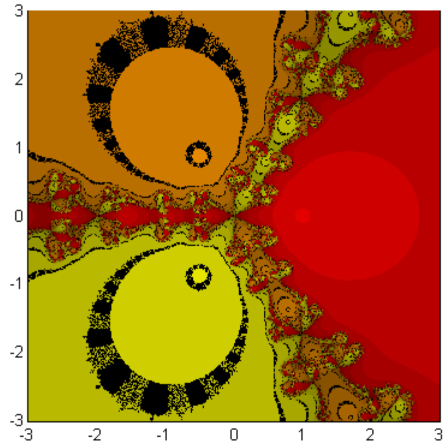

FIGURE 13. KT8. The results are for the polynomial $z^3 - 1$

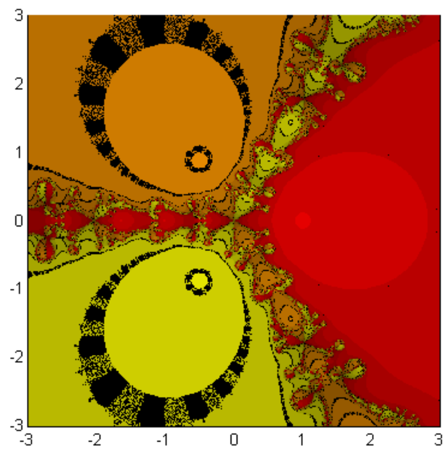

FIGURE 14. N8. The results are for the polynomial $z^3 - 1$

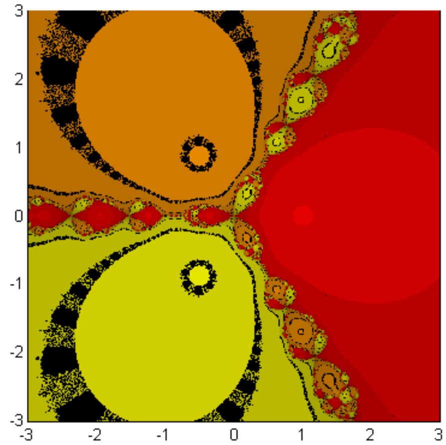

FIGURE 15. WL. The results are for the polynomial $z^3 - 1$

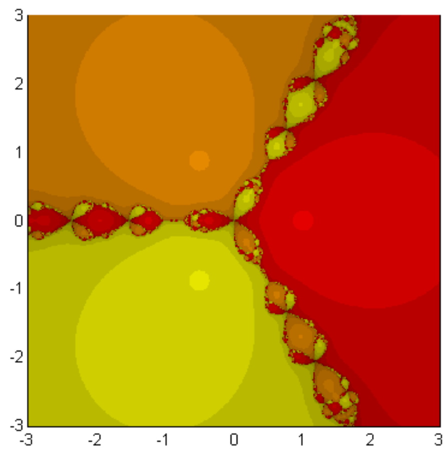

FIGURE 16. WLN. The results are for the polynomial $z^3 - 1$

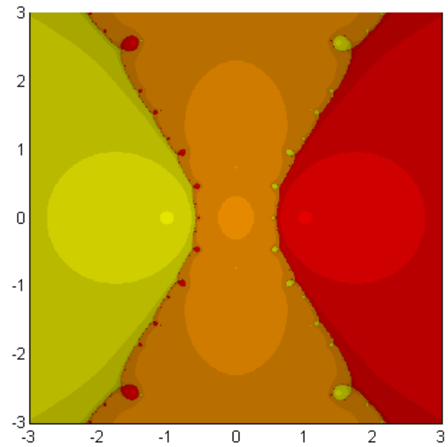

FIGURE 17. JHID8. The results are for the polynomial $z^3 - z$

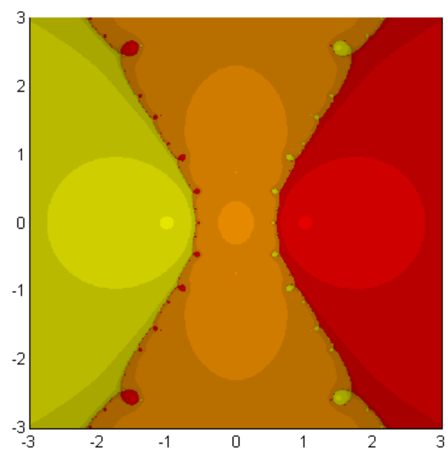

FIGURE 18. JHIF8. The results are for the polynomial $z^3 - z$

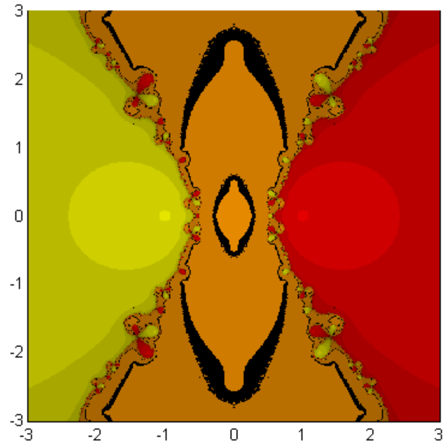

FIGURE 19. HKT. The results are for the polynomial $z^3 - z$

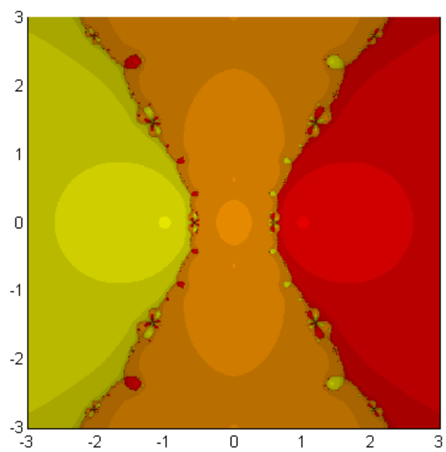

FIGURE 20. HK8. The results are for the polynomial $z^3 - z$

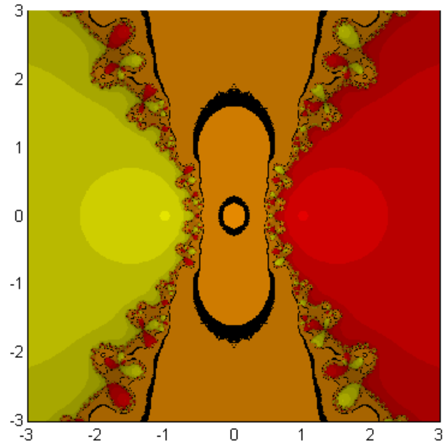

FIGURE 21. KT8. The results are for the polynomial $z^3 - z$

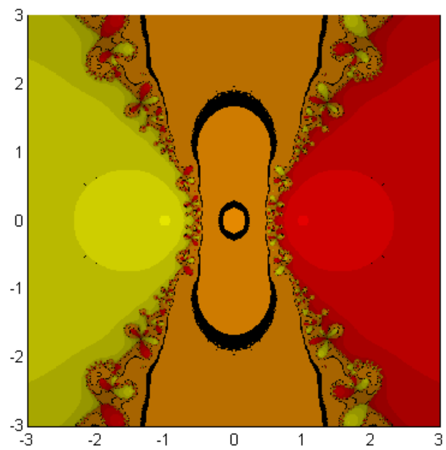

FIGURE 22. N8. The results are for the polynomial $z^3 - z$

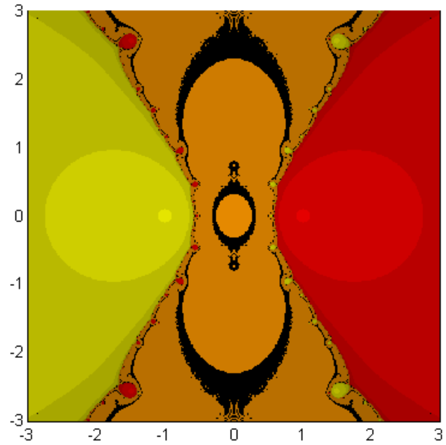

FIGURE 23. WL. The results are for the polynomial $z^3 - z$

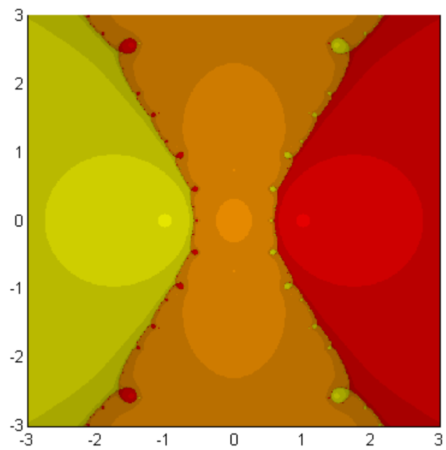

FIGURE 24. WLN. The results are for the polynomial $z^3 - z$

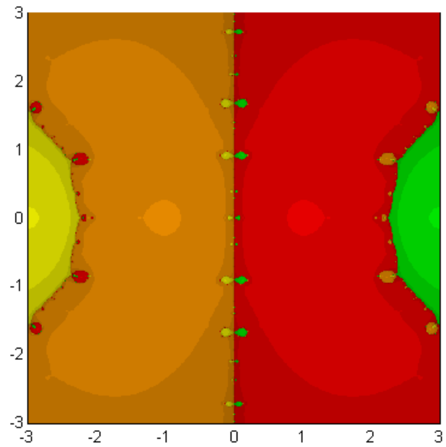

FIGURE 25. JHID8. The results are for the polynomial $z^4 - 10z^2 + 9$

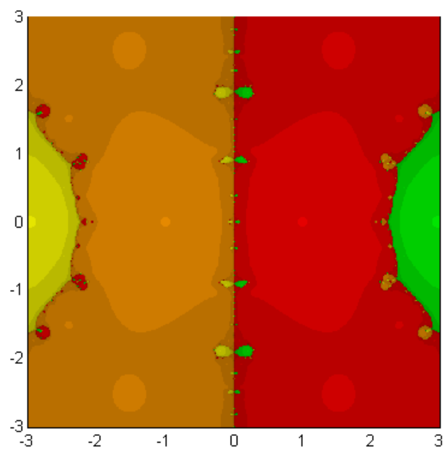

FIGURE 26. JHIF8. The results are for the polynomial $z^4 - 10z^2 + 9$

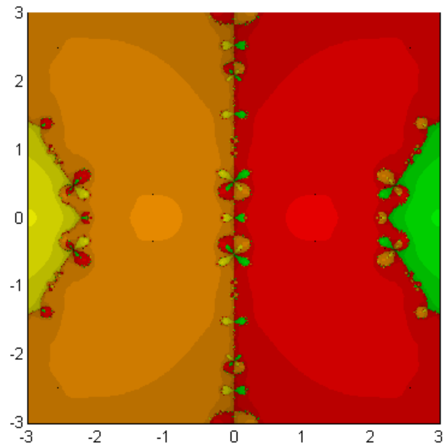

FIGURE 27. HKT. The results are for the polynomial $z^4 - 10z^2 + 9$

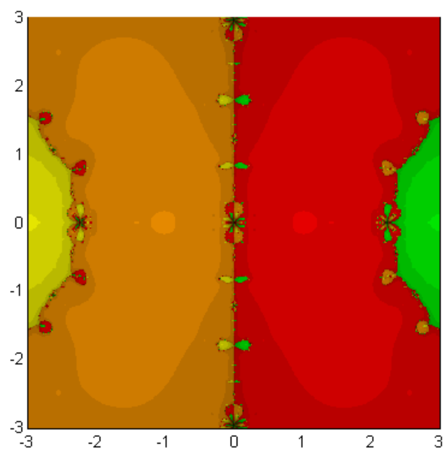

FIGURE 28. HK8. The results are for the polynomial $z^4 - 10z^2 + 9$

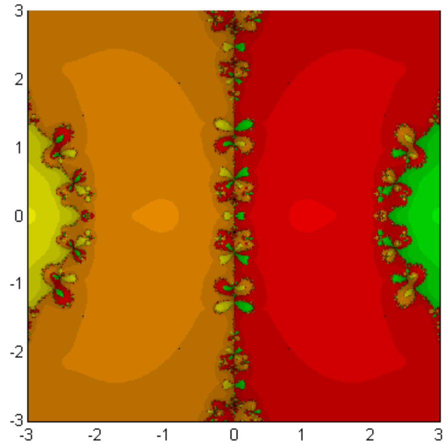

FIGURE 29. KT8. The results are for the polynomial $z^4 - 10z^2 + 9$

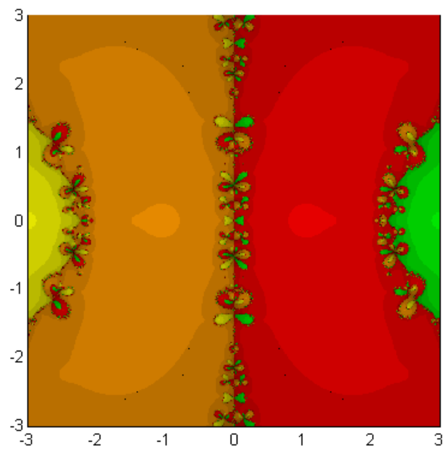

FIGURE 30. N8. The results are for the polynomial $z^4 - 10z^2 + 9$

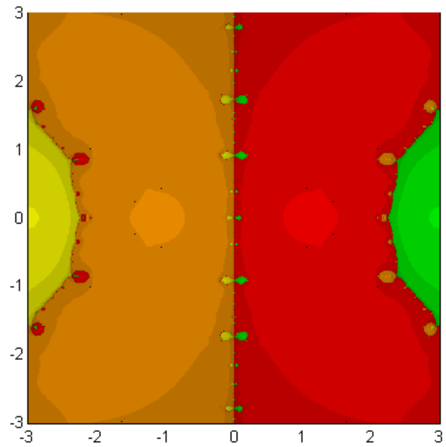

FIGURE 31. WL. The results are for the polynomial $z^4 - 10z^2 + 9$

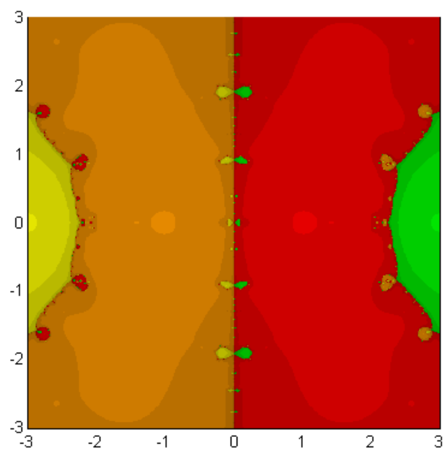

FIGURE 32. WLN. The results are for the polynomial $z^4 - 10z^2 + 9$

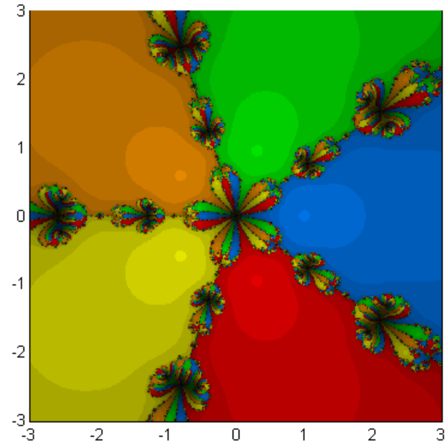

FIGURE 33. JHID8. The results are for the polynomial $z^5 - 1$

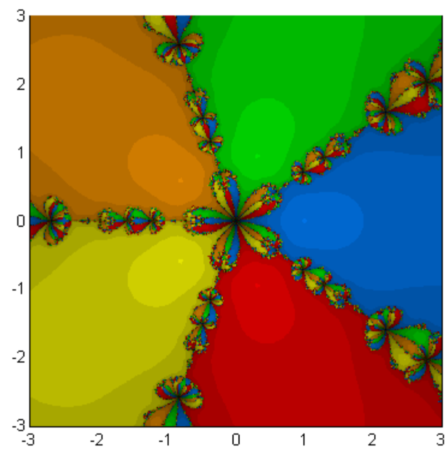

FIGURE 34. JHIF8. The results are for the polynomial $z^5 - 1$

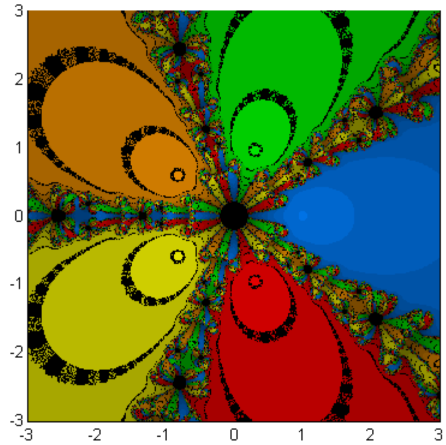

FIGURE 35. HKT. The results are for the polynomial $z^5 - 1$

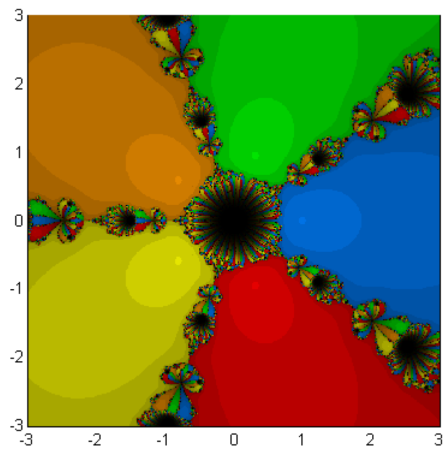

FIGURE 36. HK8. The results are for the polynomial $z^5 - 1$

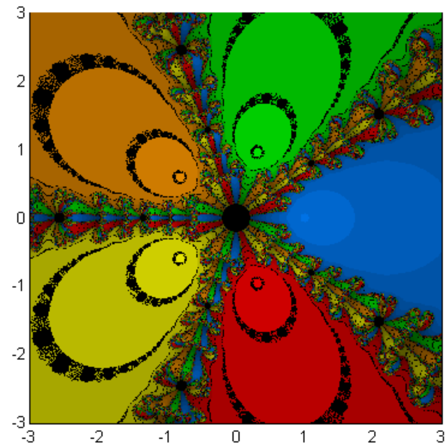

FIGURE 37. KT8. The results are for the polynomial $z^5 - 1$

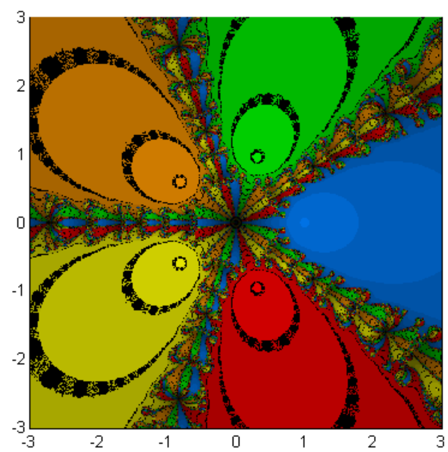

FIGURE 38. N8. The results are for the polynomial $z^5 - 1$

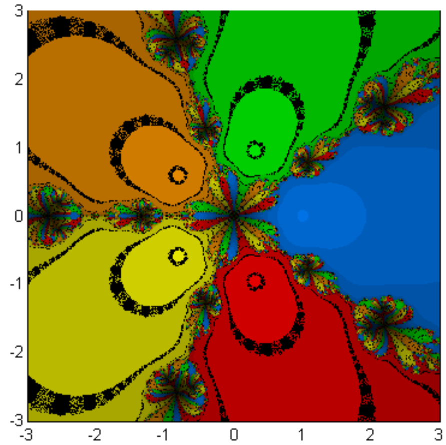

FIGURE 39. WL. The results are for the polynomial $z^5 - 1$

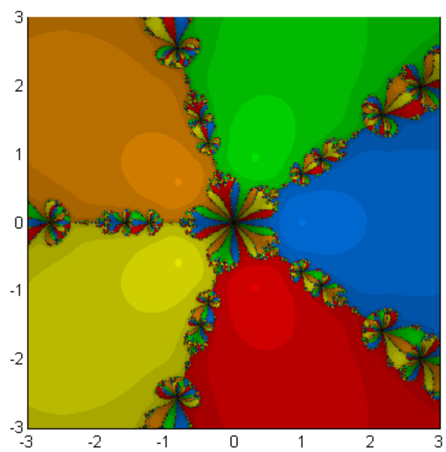

FIGURE 40. WLN. The results are for the polynomial $z^5 - 1$

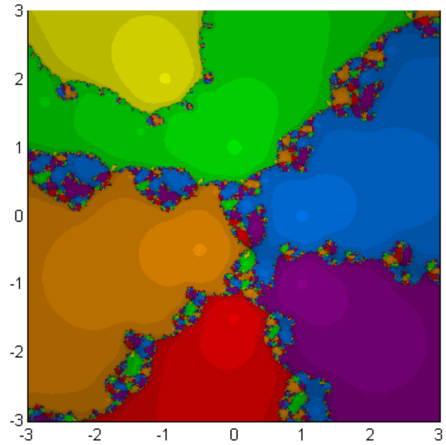

FIGURE 41. JHID8. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

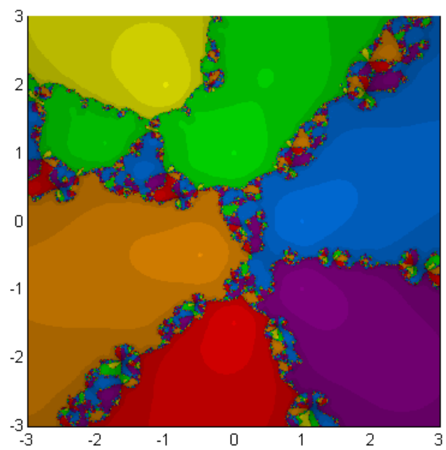

FIGURE 42. JHIF8. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

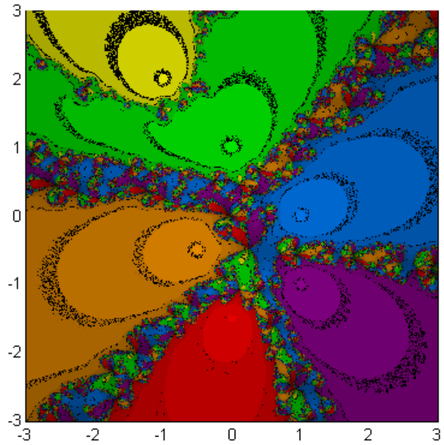

FIGURE 43. HKT. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

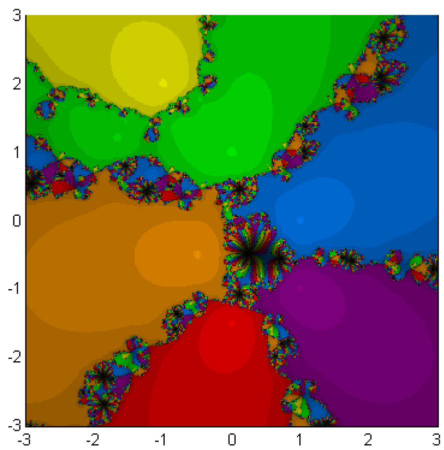

FIGURE 44. HK8. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

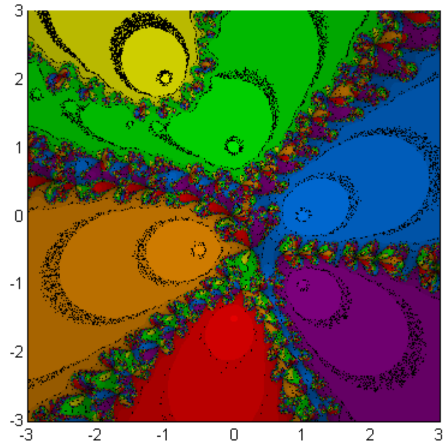

FIGURE 45. KT8. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

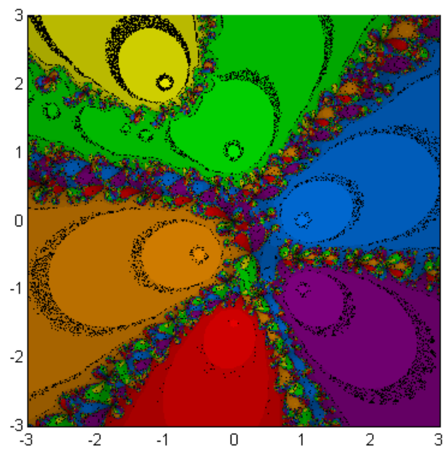

FIGURE 46. N8. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

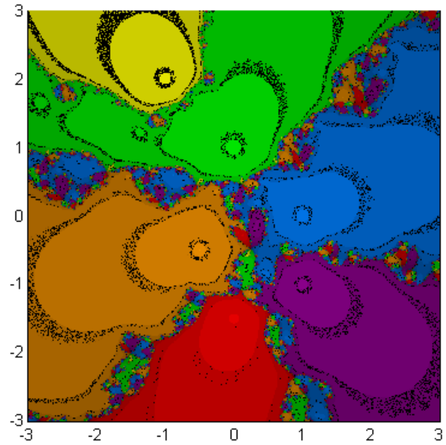

FIGURE 47. WL. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

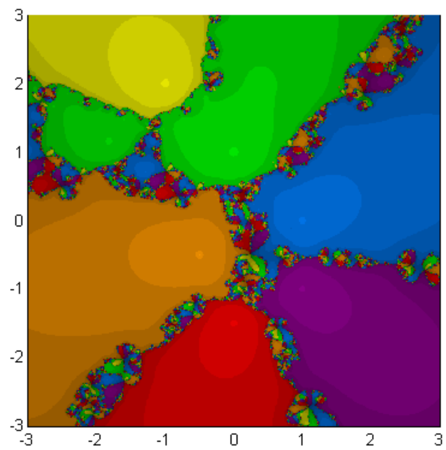

FIGURE 48. WLN. The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$

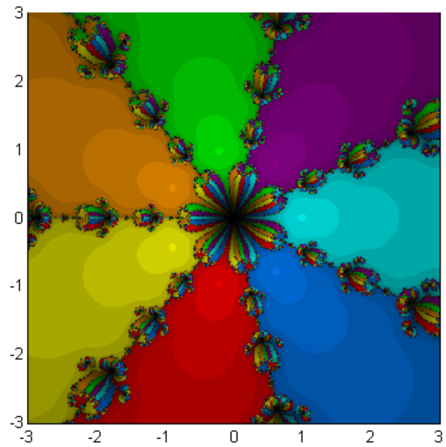

FIGURE 49. JHID8. The results are for the polynomial $z^7 - 1$

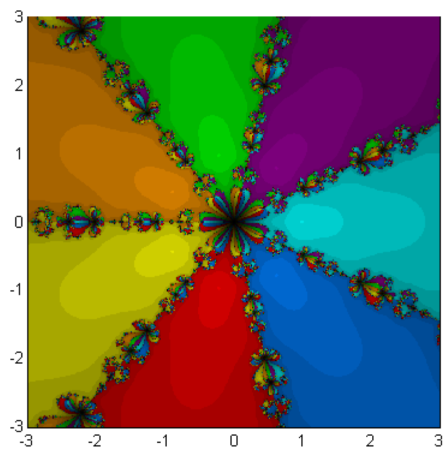

FIGURE 50. JHIF8 . The results are for the polynomial $z^7 - 1$

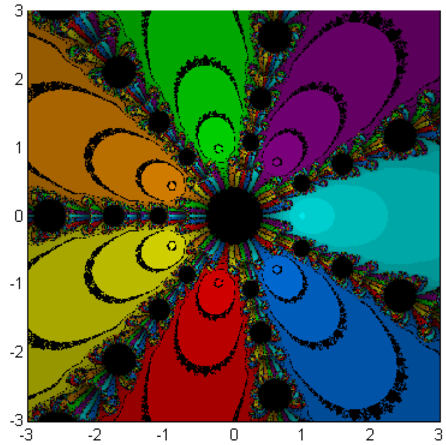

FIGURE 51. HKT. The results are for the polynomial $z^7 - 1$

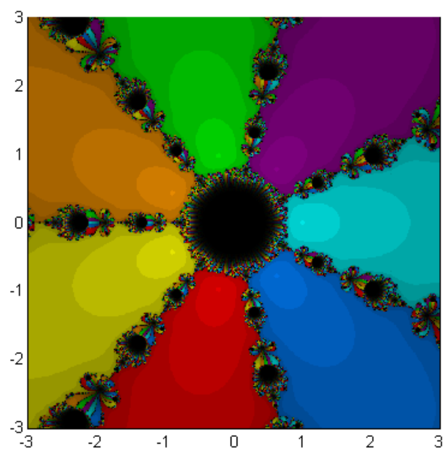

FIGURE 52. HK8. The results are for the polynomial $z^7 - 1$

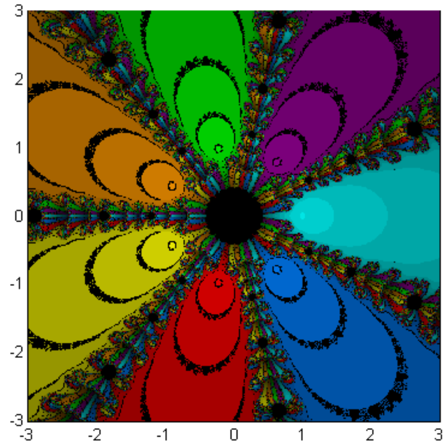

FIGURE 53. KT8. The results are for the polynomial $z^7 - 1$

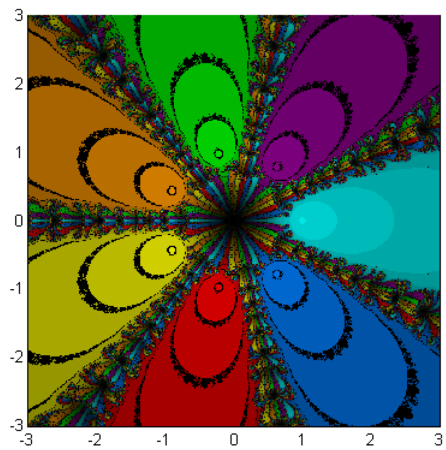

FIGURE 54. N8. The results are for the polynomial $z^7 - 1$

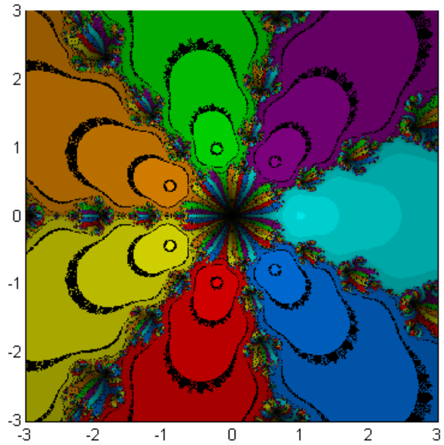

FIGURE 55. WL. The results are for the polynomial $z^7 - 1$

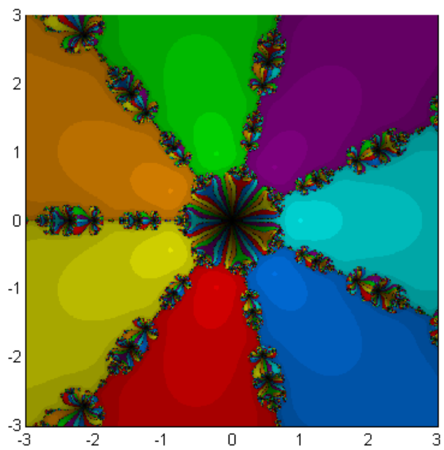

FIGURE 56. WLN. The results are for the polynomial $z^7 - 1$